Some More Functions That Are Not APN Infinitely Often. The Case of Kasami exponents

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Abstract

We prove a necessary condition for some polynomials of Kasami degree to be APN over $\mathbb{F}_{q^n}$ for large n.

1 Introduction

The vector Boolean functions are used in cryptography to construct block ciphers and an important criterion on these functions is high resistance to differential cryptanalysis.

Let $q = 2^n$ for some positive integer n. A function $f : \mathbb{F}_q \longrightarrow \mathbb{F}_q$ is said to be *almost perfect nonlinear* (APN) on $\mathbb{F}_q$ if the number of solutions in $\mathbb{F}_q$ of the equation

$$f(x+a) + f(x) = b$$

is at most 2, for all $a, b \in \mathbb{F}_q$, $a \neq 0$. Because $\mathbb{F}_q$ has characteristic 2, the number of solutions to the above equation must be an even number, for any function f on $\mathbb{F}_q$. This kind of function has a good resistance to differential cryptanalysis as was proved by Nyberg in [8].

So far, the study of APN functions has focused on power functions. Recently it was generalized to polynomials (cf. [1]).

There are many classes of function for which it can be shown that each function is APN for at most a finite number of extensions. So we fixe a finite field $\mathbb{F}_q$ and a function $f : \mathbb{F}_q \to \mathbb{F}_q$ given by a polynomial in $\mathbb{F}_q[x]$ and we set the question of whether this function can be APN for an infinite number of extensions of $\mathbb{F}_q$.

In this approach, Hernando and McGuire [5] showed a result on the classification of APN monomials which has been conjectured for 40 years: the only exponents such that the monomial x^d are APN over infinitely many

extension of $\mathbb{F}_2$ are of the form $2^i + 1$ or $4^i - 2^i + 1$. One calls these exponents *exceptional exponents*. Then it is natural to formulate for polynomial functions the following conjecture.

Conjecture 1.1 (Aubry, McGuire and Rodier) *A polynomial on $\mathbb{F}_q$ can be APN for an infinity of extensions of $\mathbb{F}_q$ only if it is CCZ equivalent (as was defined by Carlet, Charpin and Zinoviev in [4]) to a monomial x^t where t is an exceptional exponent.*

Some cases for f of small degree have been proved by the author [9]. We showed there that for some polynomial functions f which are APN on $\mathbb{F}_2^m$, the number m is bounded by an expression depending on the degree of f.

We used it for a method already used by Janwa who showed, with the help of Weil bounds, that certain cyclic codes could not correct two errors [6]. Canteaut showed by the same method that some power functions were not APN for a too large value of the exponent [3]. We were able to generalize this result to all polynomials by applying Lang-Weil's results.

Some cases of this conjecture have been studied already, in particular the case of Gold degree. We recall them in section 3. In this paper, we will study polynomials of Kasami degree. The proofs happen to be somehow the same as in Gold degree, with a few changes anyway.

2 Preliminaries

We define

$$\phi(x, y, z) = \frac{f(x) + f(y) + f(z) + f(x + y + z)}{(x + y)(x + z)(y + z)}$$

which is a polynomial in $\mathbb{F}_q[x, y, z]$. This polynomial defines a surface X in the three dimensional affine space $\mathbb{A}^3$.

If X is absolutely irreducible (or has an absolutely irreducible component defined over $\mathbb{F}_q$) then f is not APN on $\mathbb{F}_{q^n}$ for all n sufficiently large. As shown in [9], this follows from the Lang-Weil bound for surfaces, which guarantees many $\mathbb{F}_{q^n}$-rational points on the surface for all n sufficiently large.

We call $\phi_j(x, y, z)$ the ϕ function associated to the monomial x^j. The function $\phi_j(x, y, z)$ is is homogeneous of degree $j - 3$.

We recall a result due to Janwa, Wilson, [6, Theorem 5] about Kasami exponents.

Theorem 2.1 *If* $f(x) = x^{2^{2k}-2^k+1}$ *then*

$$\phi(x,y,z) = \prod_{\alpha \in \mathbb{F}_{2^k} - \mathbb{F}_2} p_\alpha(x,y,z) \tag{1}$$

where for each α*,* $p_\alpha(x,y,z)$ *is an absolutely irreducible polynomial of degree* 2^k+1 *on* $\mathbb{F}_{2^k}$ *such that* $p_\alpha(x,0,1) = (x-\alpha)^{2^k+1}$*.*

3 Some Functions That Are Not APN Infinitely Often

The best known examples of APN functions are the Gold functions x^{2^k+1} and the Kasami-Welch functions $x^{4^k-2^k+1}$. These functions are defined over $\mathbb{F}_2$, and are APN on any field $\mathbb{F}_{2^m}$ where $gcd(k,m) = 1$. For other odd degree polynomial functions, we can state a general result.

Theorem 3.1 (Aubry, McGuire and Rodier, [1]) *If the degree of the polynomial function* f *is odd and not a Gold or a Kasami-Welch number then* f *is not APN over* $\mathbb{F}_{q^n}$ *for all* n *sufficiently large.*

In the even degree case, we can state the result when half of the degree is odd, with an extra minor condition.

Theorem 3.2 (Aubry, McGuire and Rodier, [1]) *If the degree of the polynomial function* f *is* $2e$ *with* e *odd, and if* f *contains a term of odd degree, then* f *is not APN over* $\mathbb{F}_{q^n}$ *for all* n *sufficiently large.*

In [10] we have some results for the case of polynomials of degree $4e$ where e is odd.

Theorem 3.3 *If the degree of the polynomial function* f *is even such that* $\deg(f) = 4e$ *with* $e \equiv 3 \pmod 4$*, and if the polynomials of the form*

$$(x+y)(y+z)(z+x) + P$$

with

$$P(x,y,z) = c_1(x^2+y^2+z^2) + c_4(xy+xz+zy) + b_1(x+y+z) + d \tag{2}$$

for $c_1, c_4, b_1, d \in \mathbb{F}_{q^3}$*, do not divide* ϕ *then* f *is not APN over* $\mathbb{F}_{q^n}$ *for* n *large.*

We have more precise results for polynomials of degree 12.

Theorem 3.4 *If the degree of the polynomial f defined over $\mathbb{F}_q$ is 12, then either f is not APN over $\mathbb{F}_{q^n}$ for large n or f is CCZ equivalent to the Gold function x^3. In this case f is of the form*

$$L(x^3) + L_1 \text{ or } (L(x))^3 + L_1$$

where L is a linearized polynomial

$$x^4 + x^2(c^{1+q} + c^{1+q^2} + c^{q+q^2}) + xc^{1+q+q^2},$$

c is an element of $\mathbb{F}_{q^3}$ such that $c+c^q+c^{q^2} = 0$ and L_1 is a q-affine polynomial of degree at most 8 (that is a polynomial whose monomials are of degree 0 or a power of 2).

We have some results on the polynomials of Gold degree $d = 2^k + 1$.

Theorem 3.5 (Aubry, McGuire and Rodier, [1]) *Suppose $f(x) = x^d + g(x)$ where $\deg(g) \leq 2^{k-1}+1$. Let $g(x) = \sum_{j=0}^{2^{k-1}+1} a_j x^j$. Suppose moreover that there exists a nonzero coefficient a_j of g such that $\phi_j(x,y,z)$ is absolutely irreducible (where $\phi_i(x,y,z)$ denote the polynomial $\phi(x,y,z)$ associated to x^i). Then f is not APN over $\mathbb{F}_{q^n}$ for all n sufficiently large.*

4 Polynomials of Kasami Degree

Suppose the degree of f is a Kasami number $d = 2^{2k} - 2^k + 1$. Set d to be this value for this section. Then the degree of ϕ is $d - 3 = 2^{2k} - 2^k - 2$. We will prove the absolute irreducibility for a certain type of f.

Theorem 4.1 *Suppose $f(x) = x^d + g(x)$ where $\deg(g) \leq 2^{2k-1} - 2^{k-1} + 1$. Let $g(x) = \sum_{j=0}^{2^{k-1}+1} a_j x^j$. Suppose moreover that there exists a nonzero coefficient a_j of g such that $\phi_j(x,y,z)$ is absolutely irreducible. Then $\phi(x,y,z)$ is absolutely irreducible.*

Proof: Suppose $\phi(x,y,z) = P(x,y,z)Q(x,y,z)$ with $\deg P \geq \deg Q$. Write each polynomial as a sum of homogeneous parts:

$$\sum_{j=3}^{d} a_j \phi_j(x,y,z) = (P_s + P_{s-1} + \cdots + P_0)(Q_t + Q_{t-1} + \cdots + Q_0) \quad (3)$$

where P_j, Q_j are homogeneous of degree j. Then from the Theorem (2.1) we get

$$P_s Q_t = \prod_{\alpha \in \mathbb{F}_{2^k} - \mathbb{F}_2} p_\alpha(x, y, z).$$

In particular this implies that P_s and Q_t are relatively prime as the product is made of distinct irreducible factors.

The homogeneous terms of degree less than $d-3$ and greater than $2^{2k-1} - 2^{k-1}$ are 0, by the assumed bound on the degree of g. Equating terms of degree $s + t - 1$ in the equation (3) gives $P_s Q_{t-1} + P_{s-1} Q_t = 0$. Hence P_s divides $P_{s-1} Q_t$ which implies P_s divides P_{s-1} because $gcd(P_s, Q_t) = 1$, and we conclude $P_{s-1} = 0$ as $\deg P_{s-1} < \deg P_s$. Then we also get $Q_{t-1} = 0$. Similarly, $P_{s-2} = 0 = Q_{t-2}$, $P_{s-3} = 0 = Q_{t-3}$, and so on until we get the equation

$$P_s Q_0 + P_{s-t} Q_t = 0$$

since we suppose that $s \geq t$. This equation implies P_s divides $P_{s-t} Q_t$, which implies P_s divides P_{s-t}, which implies $P_{s-t} = 0$. Since $P_s \neq 0$ we must have $Q_0 = 0$.

We now have shown that $Q = Q_t$ is homogeneous. In particular, this means that $\phi_j(x, y, z)$ is divisible by $p_\alpha(x, y, z)$ for some $\alpha \in \mathbb{F}_{2^k} - \mathbb{F}_2$ and for all j such that $a_j \neq 0$. We are done if there exists such a j with $\phi_j(x, y, z)$ irreducible. Since $\phi_j(x, y, z)$ is defined over $\mathbb{F}_2$ it implies that $p_\alpha(x, y, z)$ also, which is a contradiction with the fact that α is not in $\mathbb{F}_2$.

$\square$

Remark: The hypothesis that there should exist a j with $\phi_j(x, y, z)$ is absolutely irreducible is not a strong hypothesis. This is true in many cases (see remarks in [1]). However, some hypothesis is needed, because the theorem is false without it. One counterexample is with $g(x) = x^{13}$ and $k \geq 4$ and even.

Corollary 4.1 *Suppose $f(x) = x^d + g(x)$ where g is a polynomial in $\mathbb{F}_q[x]$ such that $\deg(g) \leq 2^{2k-1} - 2^{k-1} + 1$. Let $g(x) = \sum_{j=0}^{2^{k-1}+1} a_j x^j$. Suppose moreover that there exists a nonzero coefficient a_j of g such that $\phi_j(x, y, z)$ is absolutely irreducible. Then the polynomial f is APN for only finitely many extensions of $\mathbb{F}_q$.*

4.1 On the Boundary of the First Case

If we jump one degree more we need other arguments to prove irreducibility.

Theorem 4.2 *Let $q = 2^n$. Suppose $f(x) = x^d + g(x)$ where $g(x) \in \mathbb{F}_q[x]$ and $\deg(g) = 2^{2k-1} - 2^{k-1} + 2$. Let $k \geq 3$ be odd and relatively prime to n. If $g(x)$ does not have the form $ax^{2^{2k-1}-2^{k-1}+2} + a^2x^3$ then ϕ is absolutely irreducible, while if $g(x)$ does have the form $ax^{2^{2k-1}-2^{k-1}+2} + a^2x^3$ then either ϕ is irreducible or ϕ splits into two absolutely irreducible factors which are both defined over $\mathbb{F}_q$.*

Proof: Suppose $\phi(x, y, z) = P(x, y, z)Q(x, y, z)$ with $\deg P \geq \deg Q$ and let

$$g(x) = \sum_{j=0}^{2^{2k-1}-2^{k-1}+2} a_j x^j.$$

Write each polynomial as a sum of homogeneous parts:

$$\sum_{j=3}^{d} a_j\phi_j(x, y, z) = (P_s + P_{s-1} + \cdots + P_0)(Q_t + Q_{t-1} + \cdots + Q_0).$$

Then

$$P_sQ_t = \prod_{\alpha \in \mathbb{F}_{2^k} - \mathbb{F}_2} p_\alpha(x, y, z).$$

In particular this means P_s and Q_t are relatively prime as in the previous theorem.

Since $s \geq t$, we have $s \geq 2^{2k-1} - 2^{k-1} - 1$. Comparing each degree gives $P_{s-1} = 0 = Q_{t-1}$, $P_{s-2} = 0 = Q_{t-2}$, and so on until we get the equation of degree $s + 1$

$$P_sQ_1 + P_{s-t+1}Q_t = 0$$

which implies $P_{s-t+1} = 0 = Q_1$.

If $s \neq t$ then $s \geq 2^{2k-1} - 2^{k-1}$. Note then that $a_{s+3}\phi_{s+3} = 0$. The equation of degree s is

$$P_sQ_0 + P_{s-t}Q_t = a_{s+3}\phi_{s+3} = 0.$$

This means that $P_{s-t} = 0$, so $Q_0 = 0$. We now have shown that $Q = Q_t$ is homogeneous. In particular, this means that $\phi(x, y, z)$ is divisible by

$p_\alpha(x,y,z)$ for some $\alpha \in \mathbb{F}_{2^k} - \mathbb{F}_2$, which is impossible, as we will show. Indeed, since the leading coefficient of g is not 0, the polynomial $\phi_{2^{2k-1}-2^{k-1}+2}$ occurs in ϕ; as

$$\phi_{2^{2k-1}-2^{k-1}+2} = \phi^2_{2^{2k-2}-2^{k-2}+1}(x+y)(y+z)(z+x), \tag{4}$$

this polynomial is prime to ϕ, because if $p_\alpha(x,y,z)$ occurs in the polynomials $\phi_{2^{2k-1}-2^{k-1}+2}$, then it will occur in $\phi_{2^{2k-2}-2^{k-2}+1}$. If that is the case, the polynomial $p_\alpha(x,0,1) = (x-\alpha)^{2^k+1}$ would divide $\phi_{2^{2k-2}-2^{k-2}+1}(x,0,1)$. One has

$$\begin{aligned}&(x+y)(y+z)(z+x)\phi_{2^{2k-2}-2^{k-2}+1}(x,y,z)\\ &= x^{2^{2k-2}-2^{k-2}+1} + y^{2^{2k-2}-2^{k-2}+1} + z^{2^{2k-2}-2^{k-2}+1} + (x+y+z)^{2^{2k-2}-2^{k-2}+1}\end{aligned}$$

hence

$$x(x+1)\phi_{2^{2k-2}-2^{k-2}+1}(x,0,1) = x^{2^{2k-2}-2^{k-2}+1} + 1 + (x+1)^{2^{2k-2}-2^{k-2}+1}.$$

Let $s = x - \alpha$. We have, for some polynomial R:

$$\begin{aligned} & (s+\alpha)(s+\alpha+1)s^{2^k+1} \\ = \ & (s+\alpha)^{2^{2k-2}-2^{k-2}+1} + 1 + (s+\alpha+1)^{2^{2k-2}-2^{k-2}+1} \\ = \ & \alpha^{2^{2k-2}-2^{k-2}+1} + s\alpha^{2^{2k-2}-2^{k-2}} + s^{2^{k-2}}\alpha^{2^{2k-2}-2^{k-1}+1} + 1 + \\ & \quad +(\alpha+1)^{2^{2k-2}-2^{k-2}+1} + s(\alpha+1)^{2^{2k-2}-2^{k-2}} + s^{2^{k-2}}(\alpha+1)^{2^{2k-2}-2^{k-1}+1} + s^{2^{k-2}+1}R(s). \end{aligned}$$

As $\alpha^{2^k-1} = 1$ we have $\alpha^{2^{2k-2}-2^{k-2}} = \alpha^{2^{k-2}(2^k-1)} = 1$. So

$$\begin{aligned} & (s+\alpha)(s+\alpha+1)s^{2^k+1} \\ = \ & \alpha + s + s^{2^{k-2}}\alpha^{1-2^{k-2}} + 1 + (\alpha+1) + s + s^{2^{k-2}}(\alpha+1)^{1-2^{k-2}} + s^{2^{k-2}+1}R(s) \\ = \ & s^{2^{k-2}}(\alpha^{1-2^{k-2}} + (\alpha+1)^{1-2^{k-2}}) + s^{2^{k-2}+1}R(s) \end{aligned}$$

which is a contradiction.

Suppose next that $s = t = 2^{2k-1} - 2^{k-1} - 1$ in which case the degree s equation is

$$P_sQ_0 + P_0Q_s = a_{s+3}\phi_{s+3}.$$

If $Q_0 = 0$, then

$$\phi(x,y,z) = \sum_{j=3}^{d} a_j\phi_j(x,y,z) = (P_s + P_0)Q_t$$

which implies that

$$\phi(x,y,z) = a_d\phi_d(x,y,z) + a_{2^{2k-1}-2^{k-1}+2}\phi_{2^{2k-1}-2^{k-1}+2}(x,y,z) = P_sQ_t + P_0Q_t$$

and $P_0 \neq 0$, since $g \neq 0$. So one has $\phi_{2^{2k-1}-2^{k-1}+2}$ divides $\phi_d(x,y,z)$ which is impossible by (4).

We may assume then that $P_0 = Q_0$. Then we have

$$\phi(x,y,z) = (P_s + P_0)(Q_s + Q_0) = P_sQ_s + P_0(P_s + Q_s) + P_0^2. \qquad (5)$$

Note that this implies $a_j = 0$ for all j except $j = 3$ and $j = s+3$. This means

$$f(x) = x^d + a_{s+3}x^{s+3} + a_3x^3.$$

So if $f(x)$ does not have this form, this shows that ϕ is absolutely irreducible.

If on the contrary ϕ splits as $(P_s + P_0)(Q_s + Q_0)$, the factors $P_s + P_0$ and $Q_s + Q_0$ are irreducible, as can be shown by using the same argument.

Assume from now on that $f(x) = x^d + a_{s+3}x^{s+3} + a_3x^3$ and that (5) holds. Then $a_3 = P_0^2$, so clearly $P_0 = \sqrt{a_3}$ is defined over $\mathbb{F}_q$. We claim that P_s and Q_s are actually defined over $\mathbb{F}_2$.

We know from (1) that P_sQ_s is defined over $\mathbb{F}_2$.

Also $P_0(P_s + Q_s) = a_{s+3}\phi_{s+3}$, so $P_s + Q_s = (a_{s+3}/\sqrt{a_3})\phi_{s+3}$. On the one hand, $P_s + Q_s$ is defined over $\mathbb{F}_{2^k}$ by Theorem 2.1. On the other hand, since ϕ_{s+3} is defined over $\mathbb{F}_2$ we may say that $P_s + Q_s$ is defined over $\mathbb{F}_q$. Because $(k,n) = 1$ we may conclude that $P_s + Q_s$ is defined over $\mathbb{F}_2$. Note that the leading coefficient of $P_s + Q_s$ is 1, so $a_{s+3}^2 = a_3$. Whence if this condition is not true, then ϕ is absolutely irreducible.

Let σ denote the Galois automorphism $x \mapsto x^2$. Then $P_sQ_s = \sigma(P_sQ_s) = \sigma(P_s)\sigma(Q_s)$, and $P_s + Q_s = \sigma(P_s + Q_s) = \sigma(P_s) + \sigma(Q_s)$. This means σ either fixes both P_s and Q_s, in which case we are done, or else σ interchanges them. In the latter case, σ^2 fixes both P_s and Q_s, so they are defined over $\mathbb{F}_4$. Because they are certainly defined over $\mathbb{F}_{2^k}$ by Theorem 2.1, and k is odd, they are defined over $\mathbb{F}_{2^k} \cap \mathbb{F}_4 = \mathbb{F}_2$.

Finally, we have now shown that X either is irreducible, or splits into two absolutely irreducible factors defined over $\mathbb{F}_q$. $\square$

Remark: For $k = 3$, the polynomial ϕ corresponding to $f(x) = x^{57} + ax^{30} + a^2x^3$ where $a \in \mathbb{F}_q$ is irreducible. Indeed if it were not, we would have P_{27} and Q_{27} defined over $\mathbb{F}_2$, so by Theorem 2.1 we would have $P_{27} = p_\beta(x,y,z)p_{\beta^2}(x,y,z)p_{\beta^4}(x,y,z)$ and $Q_{27} = p_{\beta^3}(x,y,z)p_{\beta^5}(x,y,z)p_{\beta^6}(x,y,z)$

for some $\beta \in \mathbb{F}_8 - \mathbb{F}_2$. So, up to inversion, we would check that $P_{27}(x, 0, 1) = (1+x+x^3)^9$ and $Q_{27}(x, 0, 1) = (1+x^2+x^3)^9$, hence $P_{27}(x, 0, 1)+Q_{27}(x, 0, 1) = (1 + x + x^3)^9 + (1 + x^2 + x^3)^9$, and one can check that this is not equal to $\phi_{30}(x, 0, 1)$ as it should be.

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